A MIXED BOUNDARY-VALUE PROBLEM FOR THE LAPLACE EQUATION

A. B. Efimov and V. N. Vorob'ev

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Using the Sommerfeld method we find the Green's function of a mixed boundary-value problem for the Laplace equation in a half-space with circular boundary conditions. A wide class of stationary problems in heat conduction, electrostatics, and elasticity theory reduce to the solution of this problem.

The method of constructing a Green's function of a mixed problem for the Laplace equation in a half-space was first given by Sommerfeld [1] for cases in which the boundary line of the boundary conditions is a straight line or two parallel lines. In this case Sommerfeld introduced a multisheeted Reimann space, whose branching line coincides with the boundary line of the boundary conditions. This approach is repeatedly used below in problems with a circular boundary line in connection with various applications [2-7]. We investigate such a boundary-value problem:

$$\begin{gathered}\Delta\varphi(x,\ y,\ z)=0, \qquad z\geqslant 0,\\ \varphi(x,\ y,\ z)\big|_{z=0}=f(x,\ y), \qquad x^2+y^2<a^2,\\ \left.\frac{\partial\varphi(x,\ y,\ z)}{\partial z}\right|_{z=0}=g(x,\ y), \qquad x^2+y^2>a^2.\end{gathered} \tag{1}$$

We consider a two-sheeted Reimann space with a circular branching line. Its Green's function for a Laplace equation has the form [2, 5, 7]

$$\begin{aligned}\omega(x,\ y,\ z,\ x_0,\ y_0,\ z_0)&=\omega(\rho,\ \theta,\ \varphi,\ \rho_0,\ \theta_0,\ \varphi_0)\\ &=\frac{1}{r}\left(\frac{1}{2}+\frac{1}{\pi}\arcsin\frac{\cos\dfrac{\theta-\theta_0}{2}}{\operatorname{ch}\dfrac{\alpha}{2}}\right),\end{aligned} \tag{2}$$

where we have introduced the toroidal coordinates:

$$\left\{\begin{aligned}\theta&=\frac{i}{2}\ln\frac{x^2+y^2+(z-ia)^2}{x^2+y^2+(z+ia)^2},\\ \rho&=\frac{1}{2}\ln\frac{(\sqrt{x^2+y^2}+a)^2+z^2}{(\sqrt{x^2+y^2}-a)^2+z^2},\\ \varphi&=\operatorname{arctg}\frac{y}{x},\end{aligned}\right.\qquad \left|\begin{aligned}x&=\frac{a\operatorname{sh}\rho}{\operatorname{ch}\rho-\cos\theta}\cos\varphi,\\ y&=\frac{a\operatorname{sh}\rho}{\operatorname{ch}\rho-\cos\theta}\sin\varphi,\\ z&=\frac{a\sin\theta}{\operatorname{ch}\rho-\cos\theta},\end{aligned}\right.$$

$$\operatorname{ch}\alpha=\operatorname{ch}\rho\operatorname{ch}\rho_0-\operatorname{sh}\rho\operatorname{sh}\rho_0\cos(\varphi-\varphi_0),$$

$$r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},$$

$\omega(\rho,\ \theta,\ \varphi,\ \rho_0,\ \theta_0,\ \varphi_0)$ is a harmonic function, single-valued in a two-sheeted Reimann space, decreasing as $1/r$, when the point $(\rho,\ \theta,\ \varphi)$ becomes infinite. As the points $(\rho,\ \theta,\ \varphi)$ and $(\rho_0,\ \theta_0,\ \varphi_0)$ approach each other, ω goes to infinity as $1/r$. In the ordinary space x, y, z this function corresponds to a two-valued function, the values of which coincide with ω on the two sheets of the Reimann space.

We take the two functions $\omega_1=\omega(\rho,\ \theta,\ \varphi,\ \rho_0,\ \theta_0,\ \varphi_0)$ and $\omega_2=\omega(\rho,\ \theta,\ \varphi,\ \rho_0,\ \theta_0+2\pi,\ \varphi_0)$, singularities

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of which are found on the different sheets of the Reimann space, but are "projected" onto a single point of ordinary space. Their difference

$$u = \omega_1 - \omega_2 = \frac{2}{\pi r} \arcsin \frac{\cos \frac{\theta - \theta_0}{2}}{\operatorname{ch} \frac{\alpha}{2}} \tag{3}$$

is [2, 5-7] a Green's function of the boundary-value problem (1) in the case f(x, y) = 0, g(x, y) ≠ 0. Actually, we can show that

$$\lim_{z \to 0} \frac{\partial u}{\partial z}\Big|_{z_0=0} = \delta(x - x_0)\,\delta(y - y_0), \quad x^2 + y^2 < a^2,$$
$$\lim_{z \to 0} u\,|_{z_0=0} = 0, \qquad x^2 + y^2 > a^2,$$

and the solution of the problem is given by the integral

$$\varphi(x, y, z) = \int_S u(x, y, z, x_0, y_0, 0)\, g(x_0, y_0)\, dx_0 dy_0, \tag{4}$$

where S is the circle $x^2 + y^2 \le a^2$.

We turn to the case f(x, y) ≠ 0, g(x, y) = 0. The Green's function of this problem is given by the equation

$$v(x, y, z, x_0, y_0) = \frac{\partial}{\partial z_0} u(x, y, z, x_0, y_0, z_0)\,|_{z_0=0}\,. \tag{5}$$

It is constructed in [2] using this method, where, however, an inaccuracy is tolerated, since instead of u, we differentiate the following difference with respect to z_0:

$$\omega(\rho, \theta, \varphi, \rho_0, \theta_0, \varphi_0) - \omega(\rho, \theta, \varphi, \rho_0, -\theta_0, \varphi_0).$$

Subsequently we obtain the general solution in the form of an integrodifferential operator [6, 10, 11].

We calculate the Green's function v:

$$v = \frac{\partial u}{\partial z_0}\Big|_{z_0=0} = \frac{2}{\pi}\left\{ \frac{z}{r_0^3} \arcsin \frac{\sqrt{R - (x^2 + y^2 + z^2 - a^2)}\cdot\sqrt{a^2 - x_0^2 - y_0^2}}{\sqrt{R(a^2 - x_0^2 - y_0^2) + (x^2 + y^2 + a^2)(x_0^2 + y_0^2 + a^2) - 4a^2(xx_0 + yy_0)}} + \frac{\sqrt{2}\, az}{r_0^2 \sqrt{(a^2 - x_0^2 - y_0^2)\,[R - (x^2 + y^2 + z^2 - a^2)]}} \right\}, \tag{6}$$

where

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2 + z^2, \qquad R = \sqrt{(x^2 + y^2 + z^2 - a^2)^2 + 4a^2 z^2}\,.$$

We can verify that

$$\lim_{z \to 0} v = \delta(x - x_0)\,\delta(y - y_0), \qquad x^2 + y^2 < a^2,$$
$$\lim_{z \to 0} \frac{\partial v}{\partial z} = 0, \qquad x^2 + y^2 > a^2,$$

and the general solution of the problem has the form

$$\varphi(x, y, z) = \int_S v(x, y, z, x_0, y_0)\, f(x_0, y_0)\, dx_0 dy_0. \tag{7}$$

In applications we frequently must determine $\partial\varphi/\partial z$ on the surface z = 0. Differentiating v, we obtain the kernel K of the operator connection $\partial\varphi/\partial z$ with f(x, y) for small z:

$$K(x, y, z, x_0, y_0) = \frac{\partial v}{\partial z}$$
$$= \frac{2}{\pi}\left\{\left(\frac{1}{r_0^3} - \frac{3z^2}{r_0^5}\right) \arcsin \frac{\sqrt{(a^2 - x^2 - y^2)(a^2 - x_0^2 - y_0^2)}}{\sqrt{[a^2 - (xx_0 + yy_0)]^2 + (xy_0 - yx_0)^2}} + \frac{a}{r_0^2 \sqrt{(a^2 - x^2 - y^2)(a^2 - x_0^2 - y_0^2)}}\right\} + O\left(\frac{z}{r_0^3}\right), \quad x^2 + y^2 < a^2. \tag{8}$$

For $z \to 0$ the kernel K has a singularity $1/r_0^3$, and the differentiated integral (7) diverges. We construct it by regularization in the usual way [8]

$$\int_S K(x, y, z, x_0, y_0) f(x_0, y_0) dx_0 dy_0$$

$$= \int_S K(x, y, z, x_0, y_0) [f(x_0, y_0) - f(x, y)] dx_0 dy_0$$

$$+ f(x, y) \int_S K(x, y, z, x_0, y_0) dx_0 dy_0. \tag{9}$$

The last integral on the right side of (9) is the well known [6, 9] solution of the problem for $f(x, y) = 1$. Now, converting to the limit for $z \to 0$, we obtain the unknown operator for $\partial\varphi/\partial z$ on the surface of the half-space:

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = f(x, y) \lim_{z \to 0} \int_S K(x, y, z, x_0, y_0) dx_0 dy_0$$

$$+ \int_S K(x, y, 0, x_0, y_0) [f(x_0, y_0) - f(x, y)] dx_0 dy_0$$

$$= \frac{f(x, y)}{\pi^2 \sqrt{a^2 - x^2 - y^2}}$$

$$+ \frac{2}{\pi} \int_S \left[\frac{1}{r_{00}^3} \arcsin \sqrt{\frac{(a^2 - x^2 - y^2)(a^2 - x_0^2 - y_0^2)}{[a^2 - (xx_0 + yy_0)]^2 + (xy_0 - yx_0)^2}} \right.$$

$$\left. + \frac{a}{r_{00}^2 \sqrt{(a^2 - x^2 - y^2)(a^2 - x_0^2 - y_0^2)}} \right] [f(x_0, y_0) - f(x, y)] dx_0 dy_0, \tag{10}$$

where $r_{00}^2 = (x - x_0)^2 + (y - y_0)^2$.

For boundedness of the integral (10) inside the circle S it is sufficient that the second partial derivatives of $f(x, y)$ be bounded.

NOTATION

ω, u, v	are the Green's functions;
δ	is the Dirac delta function;
K	is the kernel of the integral operator.

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